Period morphisms and syntomic cohomology

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Introduction: comparison theorems in p-adic Hodge theory

Let p be a prime number.
$K/\mathbb{Q}_p$ complete extension with perfect residue field k and ring of integers $\mathcal{O}_K$. Let ϖ be a uniformiser in $\mathcal{O}_K$.
Denote $C := \widehat{\overline{K}}$ and $\mathcal{O}_C$ its valuation ring.

Fontaine's rings

Let $\mathcal{O}_C^\flat := \varprojlim_{x\mapsto x^p} \mathcal{O}_C/p$ and $\mathbb{A}_{\mathrm{inf}} := W(\mathcal{O}_C^\flat)$.
Let $[\cdot] : \mathcal{O}_C^\flat \to \mathbb{A}_{\mathrm{inf}}$ be the Teichmüller lift.

For $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{O}_C^\flat$ a system of roots of unity, we consider the elements:

$$\mu = [\varepsilon] - 1, \quad t = \log(1+\mu) \quad \text{et} \quad \xi = \frac{\mu}{\varphi^{-1}(\mu)}.$$

There is a natural map $\theta : \mathbb{A}_{\mathrm{inf}} \to \mathcal{O}_C$ with kernel $\ker(\theta) = (\xi)$.

Fontaine's rings

Define $\mathbb{B}_{\mathrm{dR}}^+$ as the completion of $\mathbb{A}_{\mathrm{inf}}\left[\frac{1}{p}\right]$ with respect to the ideal $\ker(\theta)$ and let $\mathbb{B}_{\mathrm{dR}} := \mathrm{Frac}(\mathbb{B}_{\mathrm{dR}}^+)$.

- $\mathbb{B}_{\mathrm{dR}}^+$ is a DVR with residue field C.
- It is endowed with an action of G_K such that $\mathbb{B}_{\mathrm{dR}}^{G_K} = K$.
- $t \in \mathbb{B}_{\mathrm{dR}}^+$ is a uniformiser.
- There is a filtration $F^r\mathbb{B}_{\mathrm{dR}} = t^r\mathbb{B}_{\mathrm{dR}}$.

De Rham comparison theorem

Theorem

Let X be an algebraic variety smooth and proper over K. There is a natural isomorphism:

$$H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{dR}} \cong H^i_{\text{dR}}(X) \otimes_K \mathbb{B}_{\text{dR}}$$

compatible with the Galois action and the filtrations.

De Rham comparison theorem

In particular,

$$(H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{dR}})^{G_K} \cong H^i_{\text{dR}}(X_K).$$

But not enough to recover étale cohomology from de Rham cohomology.

Fontaine's rings

Let $\mathcal{O}_F := W(k)$, $F = \mathrm{Frac}(\mathcal{O}_F)$.
Define

$$\mathbb{A}_{\mathrm{cris}} := \mathbb{A}_{\mathrm{inf}}\left[\frac{\xi^k}{k!}, k \geq 0\right]^{\wedge_p} \text{ and } \mathbb{B}_{\mathrm{cris}}^+ := \mathbb{A}_{\mathrm{cris}}\left[\frac{1}{p}\right]$$

and let $\mathbb{B}_{\mathrm{cris}} := \mathbb{B}_{\mathrm{cris}}^+\left[\frac{1}{t}\right]$.

- $\mathbb{B}_{\mathrm{cris}}$ is endowed with an action of G_K such that $\mathbb{B}_{\mathrm{cris}}^{G_K} = F$.
- There is a σ-linear Frobenius morphism φ.
- There exists an embedding $\mathbb{B}_{\mathrm{cris}} \otimes_F K \to \mathbb{B}_{\mathrm{dR}}$ and we endow $\mathbb{B}_{\mathrm{cris}} \otimes_F K$ with the induced filtration.

Crystalline comparison theorem

Assume that X has good reduction i.e. there is an integral model $\mathcal{X}$ smooth and proper over $\mathcal{O}_K$.
The crystalline cohomology

$$H^i_{\mathrm{cris}}(\mathcal{X}_k/\mathcal{O}_F)_{\mathbb{Q}} := H^i_{\mathrm{cris}}(\mathcal{X}_k/\mathcal{O}_F) \otimes_{\mathcal{O}_F} F$$

is a finite dimensional F-vector space with a σ-semi-linear Frobenius morphism φ and there is an isomorphism:

$$H^i_{\mathrm{cris}}(\mathcal{X}_k/\mathcal{O}_F)_{\mathbb{Q}} \otimes_F K \cong H^i_{\mathrm{dR}}(X_K).$$

Crystalline comparison theorem

Theorem

Let X be an algebraic variety smooth and proper over K, with good reduction. There is a natural isomorphism:

$$H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{cris}} \cong H^i_{\text{cris}}(\mathcal{X}_k/\mathcal{O}_F)_{\mathbb{Q}} \otimes_F \mathbb{B}_{\text{cris}}$$

compatible with the Galois action, the action of the Frobenius and the filtration after tensoring by $\mathbb{B}_{\text{dR}}$.

Crystalline comparison theorem

In particular,

$$(H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{cris}})^{G_K} \cong H^i_{\text{cris}}(\mathcal{X}_k/\mathcal{O}_F)_{\mathbb{Q}}$$

$$H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong (H^i_{\text{cris}}(\mathcal{X}_k/\mathcal{O}_F)_{\mathbb{Q}} \otimes_F \mathbb{B}_{\text{cris}})^{\varphi=1} \cap F^0(H^i_{\text{dR}}(X_K) \otimes_K \mathbb{B}_{\text{dR}}).$$

Fontaine's rings

Define $\mathbb{B}_{\mathrm{st}} = \mathbb{B}_{\mathrm{cris}}[u]$.

- $\mathbb{B}_{\mathrm{st}}$ is endowed with an action of G_K such that $\mathbb{B}_{\mathrm{st}}^{G_K} = F$.
- We can extend the Frobenius morphism to $\mathbb{B}_{\mathrm{st}}$ by setting $\varphi(u) = pu$.
- There is a monodromy operator $N = -\frac{d}{du}$ nilpotent such that $N\varphi = p\varphi N$ and $\mathbb{B}_{\mathrm{st}}^{N=0} = \mathbb{B}_{\mathrm{cris}}$.

Semi-stable comparison theorem

Assume X has semi-stable reduction i.e. there exists an integral model $\mathcal{X}$ such that locally $\mathcal{X}$ can be written:

$$\mathrm{Spec}(R) \text{ with } R \text{ étale over } \mathcal{O}_K[X_1,\dots,X_d]/(X_1\dots X_a - \varpi).$$

Let $\mathcal{O}_F^0$ the log-scheme $\mathrm{Spec}(\mathcal{O}_F)$ endowed with the log-structure induced by $\mathbb{N} \to \mathcal{O}_{\mathrm{Spec}(\mathcal{O}_F)}, 1 \mapsto 0$.

Semi-stable comparison theorem

The Hyodo-Kato cohomology

$$H^i_{\mathrm{HK}}(X) := H^i_{\mathrm{cris}}(\mathcal{X}_k/\mathcal{O}_F^0) \otimes_{\mathcal{O}_F} F$$

is a finite dimensional F-vector space with a σ-semi-linear Frobenius morphism φ and a monodromy operator N nilpotent such that $N\varphi = p\varphi N$ and there is an isomorphism:

$$\iota_{\mathrm{HK}} : H^i_{\mathrm{HK}}(X) \otimes_F K \xrightarrow{\sim} H^i_{\mathrm{dR}}(X).$$

Semi-stable comparison theorem

Theorem

Let X be an algebraic variety smooth and proper over K, with semi-stable reduction. There exists a natural isomorphism:

$$H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{st}} \cong H^i_{\text{HK}}(X) \otimes_F \mathbb{B}_{\text{st}}$$

compatible with the Galois action, the actions of φ and N and the filtration after tensoring by $\mathbb{B}_{\text{dR}}$.

Semi-stable comparison theorem

In particular,

$$(H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{st}})^{G_K} \cong H^i_{\text{HK}}(X)$$

$$H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong (H^i_{\text{HK}}(X) \otimes_F \mathbb{B}_{\text{st}})^{\varphi=1, N=0} \cap F^0(H^i_{\text{dR}}(X_K) \otimes_K \mathbb{B}_{\text{dR}}).$$

History

- Fontaine-Messing (1987), Kato-Messing (1992), Kato (1994), Tsuji (1999): syntomic cohomology
- Faltings (1989): almost étale extensions
- Nizioł (1998): K-theory
- Beilinson (2013): h-topology
- Colmez-Nizioł (2017): syntomic cohomology
- Bhatt-Morrow-Scholze (2018), Česnavičius-Koshikawa (2019) : $\mathbb{A}_{\mathrm{inf}}$-cohomology

Proof via syntomic cohomology

In the proofs using syntomic cohomology, the period isomorphism is induced by the maps:

$$H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r)) \xleftarrow{\alpha_{i,r}} H^i_{\text{syn}}(X_{\overline{K}}, \mathbb{Q}_p(r)) \to H^i_{\text{HK}}(X) \otimes_F \mathbb{B}_{\text{st}}\,.$$

where $\alpha_{i,r}$ is an isomorphism for $i \leq r$.

Syntomic cohomology and p-adic nearby cycles

Assume that locally $\mathfrak{X}$ can be written:

$$\mathrm{Spf}(R) \text{ with } R \text{ étale over } R_\square := \mathcal{O}_K\{X_1, \ldots, X_d, \frac{1}{X_1 \ldots X_a}, \frac{\varpi}{X_{a+1} \ldots X_{a+b}}\}.$$

$\mathfrak{X}$ $\rightsquigarrow$ log-structure induced by $\mathfrak{X}_k$ and $D := \{X_{a+b+1} \ldots X_d = 0\}$

$\downarrow$ log-smooth

$\mathcal{O}_K^\times$ $\rightsquigarrow$ log-structure induced by the closed point

Denote $\overline{\mathfrak{X}} := \mathfrak{X}_{\mathcal{O}_C}$ and $\mathfrak{X}_{C,\mathrm{tr}}$ the trivial locus of $\mathfrak{X}_C$.

p-adic nearby cycles

Consider the morphisms:

$$\mathfrak{X}_{\overline{k},\text{ét}} \xrightarrow{\bar{i}} \mathfrak{X}_{\mathcal{O}_C,\text{ét}} \xleftarrow{\bar{j}} \mathfrak{X}_{C,\text{tr},\text{ét}}$$

For r and n integers, the sheaf of p-adic nearby cycles is given by:

$$\bar{i}^* R\bar{j}_* \mathbb{Z}/p^n(r)'_{\mathfrak{X}_{C,\text{tr}}} \in D(\mathfrak{X}_{\overline{k},\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$$

where $\mathbb{Z}/p^n(r)' = \frac{1}{a(r)!p^{a(r)}}\mathbb{Z}/p^n(r)$ for $r = a(r)(p-1) + b(r)$ with $0 \leq b(r) < (p-1)$.

Syntomic sheaves

For n and r integers, consider:

- $R\Gamma_{\mathrm{cris}}(\overline{\mathfrak{X}}_n) = R\Gamma_{\mathrm{cris}}(\overline{\mathfrak{X}}_n/W_n(\overline{k}), \mathcal{O}_{\overline{\mathfrak{X}}_n/W_n(\overline{k})})$, the absolute crystalline cohomology of $\overline{\mathfrak{X}}_n$,
- $J_n^{[r]}$ the r-th divided power of the sheaf of ideals $J_n = \ker(\mathcal{O}_{\overline{\mathfrak{X}}_n/W_n(\overline{k})} \to \mathcal{O}_{\overline{\mathfrak{X}}_n})$.

Syntomic sheaves

The (geometric) syntomic complexes of $\mathfrak{X}$ are given by:

$$R\Gamma_{\mathrm{syn}}(\overline{\mathfrak{X}}, r)_n := [R\Gamma_{\mathrm{cris}}(\overline{\mathfrak{X}}_n, J_n^{[r]}) \xrightarrow{p^r - \varphi} R\Gamma_{\mathrm{cris}}(\overline{\mathfrak{X}}_n)]$$

$$R\Gamma_{\mathrm{syn}}(\overline{\mathfrak{X}}, r) := \mathrm{holim}_n \, R\Gamma_{\mathrm{syn}}(\overline{\mathfrak{X}}, r)_n.$$

Definition

The syntomic sheaf $\mathcal{S}_n(r)_{\mathfrak{X}_{\mathcal{O}_C}}$ is defined as the sheaf in $D(\mathfrak{X}_{\overline{k},\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$ associated to the presheaf $\mathfrak{U} \mapsto R\Gamma_{\mathrm{syn}}(\mathfrak{U}, r)_n$.

Main theorem

Theorem 1 (G., 2020)

For all $0 \leq k \leq r$, there exists a p^N-isomorphism

$$\alpha_{r,n}^0 : \mathcal{H}^k(\mathcal{S}_n(r)_{\mathfrak{X}_{\mathcal{O}_C}}) \to \bar{i}^* R^k \bar{j}_* \mathbb{Z}/p^n(r)'_{\mathfrak{X}_{C,\mathrm{tr}}}$$

where N is an integer which depends only on r (not on $\mathfrak{X}$ nor n).

Remarks

- Tsuji's proof uses the map originally defined by Fontaine-Messing $\alpha_{r,n}^{\mathrm{FM}}$.
- Colmez and Nizioł showed that the arithmetic Fontaine-Messing map:

$$\alpha_{r,n}^{\mathrm{FM}} : \mathcal{H}^k(\mathcal{S}_n(r)_{\mathfrak{X}_{\mathcal{O}_K}}) \to i^* R^k j_* \mathbb{Z}/p^n(r)'_{\mathfrak{X}_{K,\mathrm{tr}}}$$

  is a p^N-isomorphism. To do that, they constructed a local model of $\alpha_{r,n}^{\mathrm{FM}}$.
- The local construction of $\alpha_{r,n}^0$ is the geometric version of the local morphism of Colmez and Nizioł, but here, I globalize it directly.

Semi-stable conjecture

Assume now that $\mathfrak{X}/\mathcal{O}_K$ is proper.
Using

$$R\Gamma_{\mathrm{syn}}(\mathfrak{X}_{\mathcal{O}_C}, r)_{\mathbb{Q}} \xrightarrow{\sim} [[R\Gamma_{\mathrm{HK}}(\mathfrak{X}) \otimes_F \mathbb{B}_{\mathrm{st}}^+]^{\varphi=p^r, N=0} \xrightarrow{\iota_{\mathrm{HK}}} (R\Gamma_{\mathrm{dR}}(\mathfrak{X}_K) \otimes_K \mathbb{B}_{\mathrm{dR}}^+)/F^r]$$

and the theory of Banach-Colmez spaces, we can prove that α_r^0 induces:

$$\widetilde{\alpha}^0 : H^i_{\text{ét}}(\mathfrak{X}_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{st}} \cong H^i_{\mathrm{HK}}(\mathfrak{X}) \otimes_{W(k)} \mathbb{B}_{\mathrm{st}}$$

Uniqueness of period morphisms

Let $\widetilde{\alpha}^{\mathrm{FM}}$ and $\widetilde{\alpha}^{\mathrm{CK}}$ denote the period morphisms obtained by Tsuji and Česnavičius-Koshikawa respectively, then:

Theorem 2 (G., 2020)

The morphisms $\widetilde{\alpha}^{\mathrm{FM}}$ and $\widetilde{\alpha}^{0}$ on one hand and $\widetilde{\alpha}^{\mathrm{CK}}$ and $\widetilde{\alpha}^{0}$ on the other hand, are equal. In particular, $\widetilde{\alpha}^{\mathrm{FM}} = \widetilde{\alpha}^{\mathrm{CK}}$.

Corollary

The period morphisms of Faltings, Nizioł, Beilinson and Bhatt-Morrow-Scholze (and Česnavičius-Koshikawa) are equal.

Local result

Local models

Assume $\mathfrak{X} = \mathrm{Spf}(R)$.
We want to use a simple complex representing $R\Gamma_{\mathrm{syn}}(\mathfrak{X}_{\mathcal{O}_C}, r)$.
We use:

$$R\Gamma_{\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C}) \cong R\Gamma_{\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C}/\mathbb{A}_{\mathrm{cris}})$$

(by the universal property of $\mathbb{A}_{\mathrm{cris}}$).

$\rightsquigarrow$ It suffices to work with a lifting R^+_{cris} of R over

$$\mathbb{A}_{\mathrm{cris}}\{X_1, \dots, X_d, \frac{1}{X_1 \dots X_a}, \frac{\varpi}{X_{a+1} \dots X_{a+b}}\}.$$

Local syntomic complex

Endow R^+_{cris} with the filtration coming from $\mathbb{A}_{\text{cris}}$ and with the Frobenius morphism given by $\varphi(X_i) = X_i^p$.

Then,

$R\Gamma_{\text{cris}}(\mathfrak{X}_{\mathcal{O}_C}/\mathbb{A}_{\text{cris}}) \cong \Omega^i_{R^+_{\text{cris}}}$ and $R\Gamma_{\text{cris}}(\mathfrak{X}_{\mathcal{O}_C}/\mathbb{A}_{\text{cris}}, J^{[r]}) \cong F^r\Omega^i_{R^+_{\text{cris}}}$

so the local syntomic complex is

$$\mathrm{Syn}(R^+_{\text{cris}}, r) := [F^r\Omega^\bullet_{R^+_{\text{cris}}} \xrightarrow{p^r - \varphi} \Omega^\bullet_{R^+_{\text{cris}}}].$$

Local étale complex

Let $\overline{R}$ be the maximal extension of R unramified outside D in characteristic 0 and $G_R := \mathrm{Gal}(\overline{R}[\frac{1}{p}]/R[\frac{1}{p}])$. Then (by a $K(\pi, 1)$ lemma):

$$R\Gamma(G_R, \mathbb{Z}/p^n(r)) \cong R\Gamma_{\text{ét}}(\mathrm{Sp}(R\left[\frac{1}{p}\right]) \setminus D_K, \mathbb{Z}/p^n\mathbb{Z}(r)).$$

Local theorem

Theorem

There exist p^N-quasi-isomorphisms:

$$\alpha_r^0 : \tau_{\leq r} \operatorname{Syn}(R^+_{\mathrm{cris}}, r) \xrightarrow{\sim} \tau_{\leq r} R\Gamma(G_R, \mathbb{Z}_p(r))$$

$$\alpha_{r,n}^0 : \tau_{\leq r} \operatorname{Syn}(R^+_{\mathrm{cris}}, r)_n \xrightarrow{\sim} \tau_{\leq r} R\Gamma(G_R, \mathbb{Z}/p^n(r))$$

where N is a constant which depends only on r.

Construction of α_0^r

Step 1: change of convergence

For $0 < u < v$, define

$$\mathbb{A}^{[u,v]} := \mathbb{A}_{\text{inf}}\left[\frac{p}{[\alpha]}, \frac{[\beta]}{p}\right]^{\wedge p} \text{ for } \alpha, \beta \in \mathcal{O}_C^\flat \text{ with } v(\alpha) = \frac{1}{v} \text{ and } v(\beta) = \frac{1}{u}$$

and let $R^{[u,v]} := R_{\text{cris}}^+ \widehat{\otimes}_{\mathbb{A}_{\text{cris}}} \mathbb{A}^{[u,v]}$.

We have a $p^{N_1 r}$-quasi-isomorphism:

$$\text{Syn}(R_{\text{cris}}^+, r) \xrightarrow{\sim} C(R^{[u,v]}, r) := [F^r\Omega^\bullet_{R^{[u,v]}} \xrightarrow{p^r - \varphi} \Omega^\bullet_{R^{[u,v]}}].$$

Step 2: Galois action

Write $\mathbb{A}_R^{[u,v]}$ the image of the embedding $R^{[u,v]} \hookrightarrow \mathbb{A}_{\overline{R}}^{[u,v]}$.

Let $\Gamma_R = \mathrm{Gal}(R_\infty/R) \cong \mathbb{Z}_p^d$ where R_∞ is the limit of the R_m obtained from R by taking the p^m-th roots of the coordinates.

The group G_R acts on $\mathbb{A}_R^{[u,v]}$ through Γ_R.

$\mathrm{Lie}\,\Gamma_R$ is a free $\mathbb{Z}_p$-module of rank d generated by $\nabla_i = t\partial_i$.

Step 2: Galois action

Write $\mathrm{Kos}(\varphi, \partial, F^r\mathbb{A}_R^{[u,v]}) := [\mathrm{Kos}(\partial, F^r\mathbb{A}_R^{[u,v]}) \xrightarrow{p^r-\varphi} \mathrm{Kos}(\partial, \mathbb{A}_R^{[u,v]})]$.

We have $p^{N_2 r}$-quasi-isomorphisms:

$$\begin{aligned} \tau_{\leq r} C(R^{[u,v]}, r) &\xrightarrow{\sim} \tau_{\leq r}\,\mathrm{Kos}(\varphi, \partial, F^r\mathbb{A}_R^{[u,v]}) && \text{choice of coordinates} \\ &\xrightarrow{\sim} \tau_{\leq r}\,\mathrm{Kos}(\varphi, \mathrm{Lie}\,\Gamma_R, \mathbb{A}_R^{[u,v]}(r)) && \text{mult. by } t^{r-\bullet} \\ &\xleftarrow{\sim} \tau_{\leq r}\,\mathrm{Kos}(\varphi, \Gamma_R, \mathbb{A}_R^{[u,v]}(r)) \end{aligned}$$

Step 3: change of ring in the Galois cohomology

Let $R_{\mathrm{inf}} := R_{\mathrm{inf}}^+\left[\frac{1}{p}\right]$ and let $\mathbb{A}_R$ be its image in $\mathbb{A}_{\overline{R}} := W\left(\overline{R}\left[\frac{1}{p}\right]^\flat\right)$.

There are quasi-isomorphisms:

$$
\begin{aligned}
\mathrm{Kos}(\varphi, \Gamma_R, \mathbb{A}_R^{[u,v]}) &\xrightarrow{\sim} \mathrm{Kos}(\varphi, \Gamma_R, \mathbb{A}_R) && \text{change of convergence}\\
&\xrightarrow{\sim} \mathrm{Kos}(\varphi, \Gamma_R, \mathbb{A}_{R_\infty}) && \text{decompletion}\\
&\xrightarrow{\sim} [R\Gamma(G_R, \mathbb{A}_{\overline{R}}) \xrightarrow{1-\varphi} R\Gamma(G_R, \mathbb{A}_{\overline{R}})] && \text{étale descent.}
\end{aligned}
$$

Step 4: Artin-Schreier theory

Finally, the Artin-Schreier exact sequence:

$$0 \to \mathbb{Z}_p(r) \to \mathbb{A}_{\overline{R}}(r) \xrightarrow{1-\varphi} \mathbb{A}_{\overline{R}}(r) \to 0$$

gives a quasi-isomorphism

$$R\Gamma(G_R, \mathbb{Z}_p(r)) \cong [R\Gamma(G_R, \mathbb{A}_{\overline{R}}(r)) \xrightarrow{1-\varphi} R\Gamma(G_R, \mathbb{A}_{\overline{R}}(r))].$$

Morphism α_0^r

To summarise, α_0^r is given by:

$$\begin{aligned}
\tau_{\leq r} R\Gamma_{\mathrm{syn}}(R,r) &\xrightarrow{\sim} \tau_{\leq r}\,\mathrm{Syn}(R^+_{\mathrm{cris}},r) \\
&\xrightarrow{\sim} \tau_{\leq r} C(R^{[u,v]},r) \\
&\xrightarrow{\sim} \tau_{\leq r}\,\mathrm{Kos}(\varphi,\partial,F^r\mathbb{A}_R^{[u,v]}) \\
&\xrightarrow{\sim} \tau_{\leq r}\,\mathrm{Kos}(\varphi,\Gamma_R,\mathbb{A}_R^{[u,v]}(r)) \\
&\xrightarrow{\sim} \tau_{\leq r}\,\mathrm{Kos}(\varphi,\Gamma_R,\mathbb{A}_R(r)) \\
&\xleftarrow{\sim} \tau_{\leq r} R\Gamma(G_R,\mathbb{Z}_p(r)).
\end{aligned}$$

Remark: arithmetic case

In the arithmetic case :

- Work with a lifting R_{ϖ}^{+} log-smooth over
$$\mathcal{O}_F\{X_0, X_1, \ldots, X_d, \frac{1}{X_1 \ldots X_a}, \frac{X_0}{X_{a+1} \ldots X_{a+b}}\}$$
with X_0 "arithmetic variable" ($X_0 \mapsto \varpi$).
- Convergence rings:
  - $r_{\varpi}^{+} := \mathcal{O}_F[\![X_0]\!]$,
  - $r_{\varpi}^{[u,v]}$ analytic functions over F convergent on the annulus $\frac{v}{e} \geq v_p(X_0) \geq \frac{u}{e}$.

Remark: arithmetic case

- The construction of the morphism is more complicated than in the geometric case:
  - Change of Frobenius (two different embeddings in the period ring $\mathbb{A}_{\overline{R}}^{[u,v]}$).
  - $\operatorname{Lie}\Gamma_R$ is not commutative.