Syntomic cohomology and p-adic nearby cycles

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Motivation

Let K be a complete extension of $\mathbb{Q}_p$ with perfect residue field k and discrete valuation ring $\mathcal{O}_K$, and let ϖ be a uniformizer of $\mathcal{O}_K$. Write $F = \mathrm{Frac}(W(k))$, $\overline{K}$ an algebraic closure of K and $G_K = \mathrm{Gal}(\overline{K}/K)$. Finally, let X be a proper scheme over $\mathcal{O}_K$.
The conjecture of Fontaine-Jannsen compares two cohomology theories attached to X:

- the p-adic étale cohomology $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$, which is a finite dimensional $\mathbb{Q}_p$-vs endowed with a continuous G_K-action,
- the Hyodo-Kato cohomology $H^i_{\mathrm{HK}}(X)$ of X which is a finite dimensional F-vs endowed with a semi-linear Frobenius φ, a monodromy operator N (with $N\varphi = p\varphi N$) and filtration after tensoring by K (via $H^i_{\mathrm{HK}}(X) \otimes_F K \xrightarrow{\sim} H^i_{\mathrm{dR}}(X_K)$.)

Explicitly, if $\mathbb{B}_{\mathrm{dR}}$ and $\mathbb{B}_{\mathrm{st}}$ are the period rings of Fontaine, the conjecture states that:

Theorem 0.1. *There is a $\mathbb{B}_{\mathrm{st}}$-linear isomorphism:*

$$\tilde{\alpha} : H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{st}} \xrightarrow{\sim} H^i_{\mathrm{HK}}(X) \otimes_F \mathbb{B}_{\mathrm{st}}. \quad (1)$$

Moreover, this isomorphism preserves the actions of G_K, φ, and N, and the filtration after tensoring by $\mathbb{B}_{\mathrm{dR}}$.

Many definitions of the isomorphism $\tilde{\alpha}$ have been given. The conjecture was fully proved by Tsuji in [4] using the Fontaine-Messing period map ([3]). An alternative period morphism was constructed by Colmez-Nizioł ([2]). These two proofs rely on a comparison theorem between syntomic sheaves and p-adic nearby cycles. More precisely, the isomorphism $\tilde{\alpha}$ is induced by maps

$$H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r)) \xleftarrow{\alpha_{i,r}} H^i_{\mathrm{syn}}(X_{\overline{K}}, \mathbb{Q}_p(r)) \to$$
$$F^r(\mathbb{B}_{\mathrm{dR}} \otimes_K H^i_{\mathrm{dR}}(X_K)) \cap (\mathbb{B}_{\mathrm{st}} \otimes_F H^i_{\mathrm{HK}}(X))^{\varphi = p^r, N=0}$$

and the main result is that $\alpha_{i,r}$ is an isomorphism for $i \leq r$.
The proof of [2] is more general since the result is established in the arithmetic case (i.e. for X_K instead of $X_{\overline{K}}$). We present here the geometric version of the construction of Colmez-Nizioł.

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Write $C = \widehat{\overline{K}}$ the completion of $\overline{K}$. Let $\mathfrak{X}$ be a log-formal scheme over $\mathrm{Spf}(\mathcal{O}_K)$ with semi-stable reduction. Here $\mathrm{Spf}(\mathcal{O}_K)$ is endowed with the logarithmic structure induced by its closed point and $\mathfrak{X}$ with the one given by the special fiber and the divisor at infinity, so that $\mathfrak{X}$ is log-smooth over $\mathcal{O}_K$. Let $\overline{\mathfrak{X}}$ denote the base change $\mathfrak{X} \otimes C$, $\mathfrak{X}_n$ the reduction of $\mathfrak{X}$ modulo p^n and $\mathfrak{X}_{\mathrm{tr}}$ the locus where the logarithmic structure is trivial.

Syntomic sheaves
For n and r integers, consider:

- $R\Gamma_{\mathrm{cris}}(\overline{\mathfrak{X}}_n)$, the absolute crystalline cohomology of $\overline{\mathfrak{X}}$,
- $J_n^{[r]}$ the r-th divided power of the ideal $J_n = \ker(\mathcal{O}_{\overline{\mathfrak{X}}_n/W_n(\bar{k})} \to \mathcal{O}_{\overline{\mathfrak{X}}_n})$ where $\mathcal{O}_{\overline{\mathfrak{X}}_n/W_n(\bar{k})}$ is the crystalline structural sheaf.

Then the (geometric) syntomic complexes of $\mathfrak{X}$ are given by the homotopy fibers:

$$R\Gamma_{\mathrm{syn}}(\overline{\mathfrak{X}}, r)_n := [R\Gamma_{\mathrm{cris}}(\overline{\mathfrak{X}}, J_n^{[r]}) \xrightarrow{p^r - \varphi} R\Gamma_{\mathrm{cris}}(\overline{\mathfrak{X}}_n)]$$
$$R\Gamma_{\mathrm{syn}}(\overline{\mathfrak{X}}, r) := \mathrm{holim}_n\, R\Gamma_{\mathrm{syn}}(\overline{\mathfrak{X}}, r)_n$$

(where φ is the Frobenius morphism).

Definition 0.2. The syntomic sheaf $\mathcal{S}_n(r)_{\overline{\mathfrak{X}}}$ is defined as the sheaf of $D(\mathfrak{X}_{\bar{k}}, \mathbb{Z}/p^n\mathbb{Z})$ associated to the presheaf $\mathfrak{U} \mapsto R\Gamma_{\mathrm{syn}}(\mathfrak{U}, r)_n$.

p-adic nearby cycles
For $\bar{i}$ and $\bar{j}$ defined as follows:

$$\begin{array}{ccccc} \mathfrak{X}_{\bar{k}} & \xhookrightarrow{\bar{i}} & \overline{\mathfrak{X}} & \xhookleftarrow{\bar{j}} & \mathfrak{X}_{\mathrm{tr},C} \\ \downarrow & & \downarrow & & \downarrow \\ \bar{k} & \longrightarrow & \mathcal{O}_C & \longleftarrow & C \end{array}$$

and r and n integers, the sheaf of p-adic nearby cycles is given by:

$$\bar{i}^* R\bar{j}_* \mathbb{Z}/p^n(r)'_{\mathfrak{X}_{\mathrm{tr},C}} \in D(\mathfrak{X}_{\bar{k}}, \mathbb{Z}/p^n\mathbb{Z})$$

where $\mathbb{Z}/p^n(r)' = \frac{1}{a(r)!p^{a(r)}}\mathbb{Z}/p^n(r)$ for $r = a(r)(p-1) + b(r)$ with $0 \leq b(r) < (p-1)$.

Theorem 0.3. *For all $0 \leq k \leq r$, there exists a p^N-isomorphism (i.e. a morphism whose kernel and cokernel are killed by p^N):*

$$\alpha^0_{r,n} : \mathcal{H}^k(\mathcal{S}_n(r)_{\overline{\mathfrak{X}}}) \to \bar{i}^* R^k \bar{j}_* \mathbb{Z}/p^n(r)'_{\mathfrak{X}_{\mathrm{tr},C}}$$

with N an integer which only depends on p and r (and not on $\mathfrak{X}$ or n).

Local result

Take $\overline{\mathfrak{X}} = \mathrm{Spf}(R)$ with R the completion of an étale algebra over $R_\square := \mathcal{O}_C\{X_1, \ldots, X_d, \frac{1}{X_1 \ldots X_a}, \frac{\varpi}{X_{a+1} \ldots X_{a+b}}\}$ (for $a + b + c = d$). Let's write $\overline{R}$ the maximal extension of R such that $\overline{R}[\frac{1}{p}]/R[\frac{1}{p}]$ is unramified outside $\{X_{a+b+1} \ldots X_d = 0\}$ and $G_R := \mathrm{Gal}(\overline{R}[\frac{1}{p}])/R[\frac{1}{p}])$. By étaleness, there is a lifting R^+_{cris} of R over $\mathbb{A}_{\mathrm{cris}} \otimes R_\square$. Then the syntomic complex $\mathrm{Syn}(R^+_{\mathrm{cris}}, r)$ is given by the homotopy fiber:

$$\mathrm{Syn}(R^+_{\mathrm{cris}}, r) := [F^r\Omega^\bullet_{R^+_{\mathrm{cris}}} \xrightarrow{p^r - \varphi} \Omega^\bullet_{R^+_{\mathrm{cris}}}]$$

where φ is the Frobenius on R^+_{cris} induced by the one on $\mathbb{A}_{\mathrm{cris}}$ and such that $\varphi(X_i) = X_i^p$. Denote by $\mathrm{Syn}(R^+_{\mathrm{cris}}, r)$ its reduction modulo p^n.

Theorem 0.5. *There exist p^N-quasi-isomorphisms (and N only depends on r):*

$$\alpha^0_r : \tau_{\leq r}\,\mathrm{Syn}(R^+_{\mathrm{cris}}, r) \xrightarrow{\sim} \tau_{\leq r} R\Gamma(G_R, \mathbb{Z}_p(r))$$

$$\alpha^0_{r,n} : \tau_{\leq r}\,\mathrm{Syn}(R^+_{\mathrm{cris}}, r)_n \xrightarrow{\sim} \tau_{\leq r} R\Gamma(G_R, \mathbb{Z}/p^n(r)).$$

Proof of (0.5)

The first step is to "change the convergence" of the elements of R^+_{cris}. In fact, there is a ring $R^{[u,v]}$ (which can be seen as a ring of analytic functions that converge on an annulus $[u, v]$), such that there is a natural $p^{N_0 r}$ quasi-isomorphism

$$\mathrm{Syn}(R^+_{\mathrm{cris}}, r) \to \mathrm{Syn}(R^{[u,v]}, r)$$

for a constant $N_0 \in \mathbb{N}$. After a choice of coordinates, the two complexes can be written as Koszul complexes. Now, using properties of $R^{[u,v]}$, we can get rid of the filtration $F^r\Omega^\bullet_{R^{[u,v]}}$ by multiplying by a suitable power of t (where $t \in \mathbb{A}_{\mathrm{cris}}$ is the element $\log(\varepsilon)$ for ε a system of primitive roots of the unity). The truncation by $\tau_{\leq r}$ is needed here.
Finally, almost étale descent and decompletion arguments show that the previous complex computes the wanted Galois cohomology.

Uniqueness

A different construction of the period isomorphism (1) is given in [1], using the $\mathbb{A}_{\mathrm{inf}}$-cohomology. Let $\widetilde{\alpha}^{\mathrm{CK}}$ denote this period morphism, $\widetilde{\alpha}^{\mathrm{FM}}$ denote the one of Fontaine-Messing and $\widetilde{\alpha}^0$ the one induced by (0.3). Then, comparing the local definitions of these maps, we get the following uniqueness result:

Theorem 0.4. *The morphisms $\widetilde{\alpha}^{\mathrm{FM}}$ and $\widetilde{\alpha}^0$ on one hand, and $\widetilde{\alpha}^{\mathrm{CK}}$ and $\widetilde{\alpha}^0$ on the other hand, are equal. In particular, $\widetilde{\alpha}^{\mathrm{FM}} = \widetilde{\alpha}^{\mathrm{CK}}$.*

References

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